QP Code: 22/PT/13/IXB(i)Old

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-9B(i) : TOPOLOGICAL GROUP

(Old Syllabus)

(Pure Mathematics)

(Spl. Paper)

(Up to July 2016 Enrolment Session)

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Symbols have their usual meanings)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) If U is a neighbourhood of the identity e of a topological group G, then prove that there is a neighbourhood V of e such that $V^2 \subseteq U$.

   b) Let G be an infinite group with co-finite topology. Is G a topological group ? Justify.

   c) Let H be a subgroup of a topological group G. Prove that the quotient group G/H is T_1 if H is closed in G.

   d) Let G and H be two topological groups and $f: G \to H$ be a homomorphism. Prove that f is continuous if f is continuous at the identity e of G.

   e) Let G be a Hausdorff locally compact abelian topological group and G^* its dual topological group. If G is discrete, prove that G^* is compact.

   f) Let G be the set of all invertible elements in a Banach algebra X. Prove that the set Z of all topological divisors of zero in X is contained in $X \setminus G$.

   g) For a Banach algebra X and for the identity operator $I: X \to X$, find $\sigma(I)$.

2. a) For a subset A of a topological group G, prove that $\overline{A} = \bigcap_{U_\alpha \in \eta_e} AU_\alpha$, where η_e denotes the neighbourhood system of the identity e of G. 5

   b) Let G be a topological group. Suppose U is any neighbourhood of the identity e of G and F a compact subset of G. Prove that there is a neighbourhood V of e such that $xV_x^{-1} \subseteq U$, for all $x \in F$. 5

3. a) Let H be a subgroup of a topological group G. If for some neighbourhood U of the identity e of G, $\overline{U} \cap H$ is closed in G, prove that H is also closed in G. 5

   b) Let U be an open neighbourhood of the identity e in a topological group G and C a compact subset of G. Prove that there is an open neighbourhood V of e such that $CVC^{-1} \subseteq U$. 5

4. a) Let G be a topological group and H a closed normal subgroup of G. Prove that the quotient group G/H with the quotient topology is a topological group. 5

   b) Prove that a topological group G is locally compact if its identity e has a compact neighbourhood. 5

5. a) Let $\{G_\alpha : \alpha \in \wedge\}$ be a family of topological groups. If $G = \prod_{\alpha \in \wedge} G_\alpha$ is the direct product of G_α's endowed with the product topology. Prove that G in a topological group. 5

   b) Give an example of a locally compact, Hausdorff topological group G which is not compact. 5

6. a) Let X be a Banach algebra with identity e and G the set of all invertible elements of X. Prove that the mapping : $G \to G$ given by $x \to x^{-1} (x \in G)$ is a continuous map. 5

   b) Let X be a complex Banach algebra with identity e. Prove that the spectrum $\sigma(x)$ of an element $x \in X$ is a compact set of scalars. 5

7. a) Prove the spectral radius formula :

   $$\gamma_\sigma(x) = \lim_{n \to \infty} \left\| x^n \right\|^{1/n}.$$ 5

   b) Let M be a maximal ideal in a commutative Banach algebra X with identity. Prove that X/M is a Banach algebra. 5